BAULKHAM HILLS HIGH SCHOOL

2015
YEAR 12
TERM 2 ASSESSMENTS

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work

Total marks – 45

- Attempt Questions 1 – 4
- All questions are NOT of equal value

Total marks – 45
Attempt Questions 1 – 4
All questions are NOT of equal value

Answer each question on the appropriate answer sheet. Each answer sheet must show your BOS#. Extra paper is available.

Your responses should include relevant mathematical reasoning and/or calculations.

Marks

<u>Question 1</u> ***(12 marks)*** Use a *separate* piece of paper

a) (i) Between which two integers does $\sqrt{5}$ lie? *1*

(ii) Using repeated applications of the "halving the interval" method, to approximate $\sqrt{5}$, correct to one decimal place. *2*

b) Use the given substitution to evaluate the following integrals.

(i) $\int 6x\sqrt{9-x^2}\ dx$ using $u = 9 - x^2$ *3*

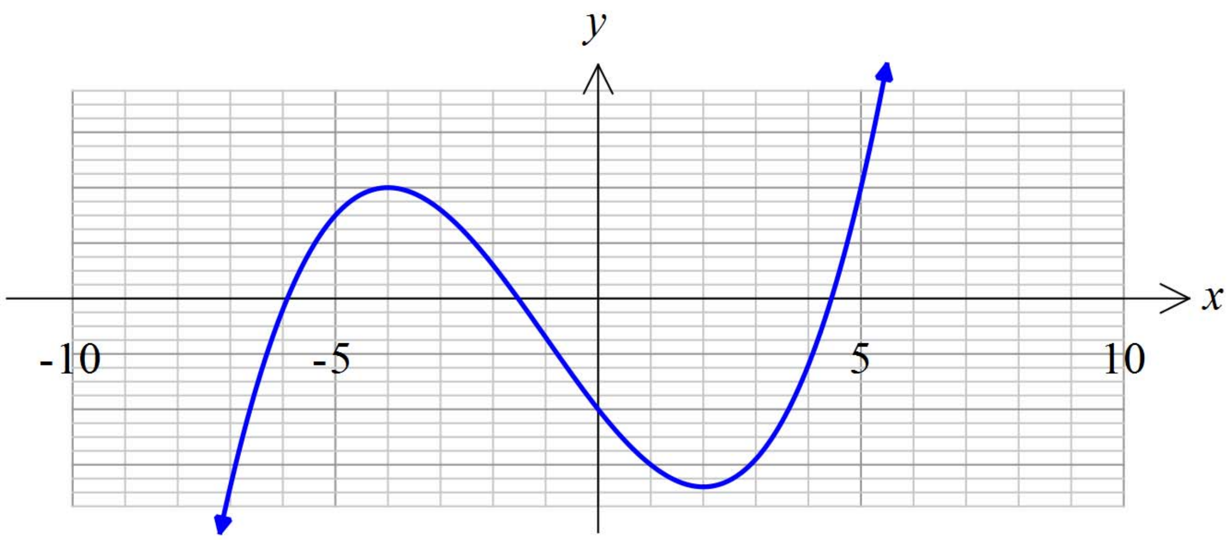
(ii) $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$ using $x = 3\sin\theta$ *3*

(iii) $\int \frac{-\sin 2x}{2 + 3\cos^2 x}\,dx$ using $u = 2 + 3\cos^2 x$ *3*

Marks

Question 2 ***(9 marks)*** Use a *separate* piece of paper

a) Below is a graph of $y = x^3 + 3x^2 - 24x - 40$.



(i) Explain why $x = 0$ would not be a good first approximation in order to find the positive solution to $x^3 + 3x^2 - 24x - 40 = 0$ using Newton's Method. *1*

(ii) Give an example of an approximation to $x^3 + 3x^2 - 24x - 40 = 0$ that would fail to find any solution using Newton's Method, and explain why it would fail. *2*

(iii) Using Newton's Method, find the positive solution to $x^3 + 3x^2 - 24x - 40 = 0$, correct to two decimal places. *3*

b) If $\ddot{x} = e^{-x}$ and initially the particle is observed to be at $x = 0$ with a velocity of 2 m/s, find v^2 as a function of x. *3*

Marks

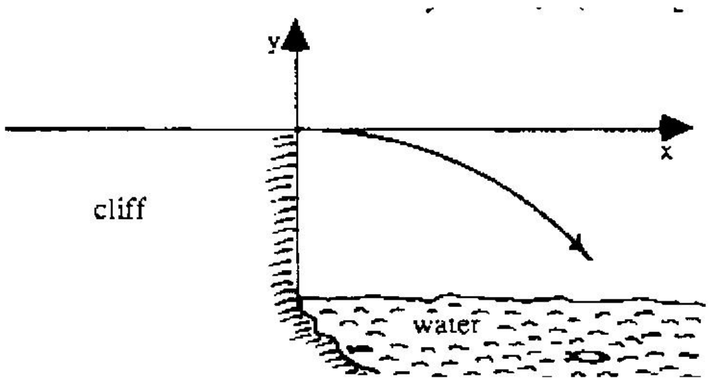
<u>Question 3</u> ***(12 marks)*** Use a *separate* piece of paper

a) The velocity of a particle moving along the x-axis is given by $v^2 = 36 - 6x - 2x^2$, where x is in metres.

(i) Prove that the particle is moving in Simple Harmonic Motion. *2*

(ii) Find the path that the particle travels. *2*

b) An object is projected horizontally from the top of a vertical cliff 40 metres above sea level with a velocity of 40 m/s. (Take $g = 10$ m/s ^{2})



(i) Using the top edge of the cliff as the origin, prove that the equations of motion are given by; *3*

$$x = 40t \quad \text{and} \quad y = -5t^2$$

(ii) Calculate when and where the object hits the water. *2*

(iii) Find the speed of the object and the angle it makes with the water, the instant it hits the water. *3*

Question 4 ***(12 marks)*** Use a *separate* piece of paper

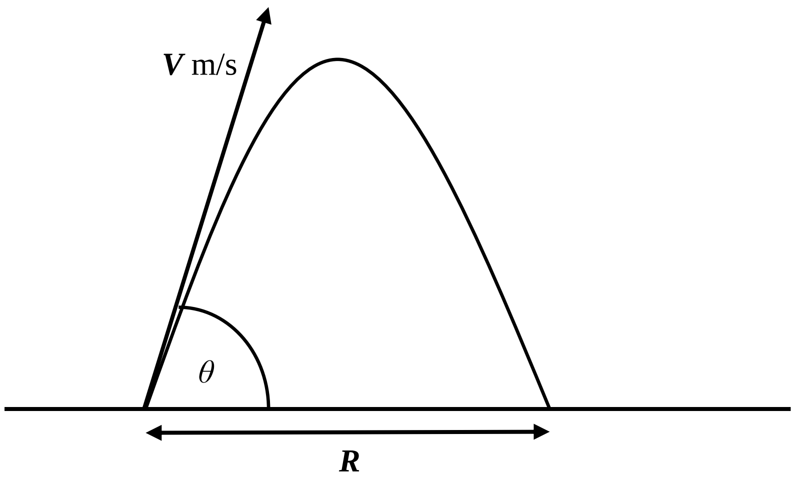
a) A particle is travelling in simple harmonic motion such that its displacement x metres from the origin is given by $\ddot{x} = -4x$.

(i) Show that $x = A\cos(2t + \beta)$ is a possible equation of motion for the particle, where A and β are positive constants. *2*

(ii) The particle is initially observed to have a velocity of 2 m/s and a displacement from the origin of 4 metres. Show that the amplitude of the motion is $\sqrt{17}$ metres. *2*

(iii) Determine the maximum speed of the particle. *1*

b) Sudarshan is in a pool holding a hose at an angle of θ to the surface of the water and water is leaving the hose with a velocity of Vm/s. The water from the hose hits the surface of the water at a distance R metres from Sudarshan.



Using the equations of motion;

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{1}{2}gt^2$$ (Do NOT prove this)

(i) Show that $R = \dfrac{V^2 \sin 2\theta}{g}$ *2*

(ii) Explain why the maximum range occurs when $\theta = 45°$ *1*

(iii) Sudarshan moves the hose so that the angle θ changes at a constant rate $\dfrac{d\theta}{dt} = k$, from an initial value of $\theta = 30°$. *2*

Prove that $\dfrac{dR}{dt} = \dfrac{2kV^2 \cos 2\theta}{g}$

(iv) Find $\dfrac{d^2R}{dt^2}$ and hence show that the motion is simple harmonic. *2*

~ END OF EXAMINATION ~

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \ n \neq -1; x \neq 0, \text{ if } \ n < 0$$

$$\int \frac{1}{x} dx = \ln x, \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \ a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \ \ x > a > 0$$

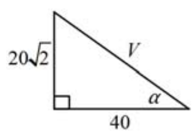
$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log x, \quad x > 0$

BAULKHAM HILLS HIGH SCHOOL
YEAR 12 EXTENSION 1 TERM 2 ASSESSMENT 2015 SOLUTIONS

Solution	Marks	Comments
QUESTION 1		
1 (a) (i) $\sqrt{5}$ lies in between 2 and 3	1	**1 mark** • Correct answer
1 (a) (ii) $f(x)=x^2-5=0$ $f(2)=2^2-5=-1<0$ $\quad f(3)=3^2-5=4>0$ $x_1=\frac{2+3}{2}=2.5$ $\quad x_2=\frac{2+2.5}{2}=2.25$ $\quad x_3=\frac{2+2.25}{2}=2.13$ $f(2.5)=(2.5)^2-5=1.25>0$ $\quad f(2.25)=(2.25)^2-5=0.0625>0$ $\quad f(2.13)=(2.13)^2-5=-0.44631<0$ $\therefore 2<\sqrt{5}<2.5$ $\quad \therefore 2<\sqrt{5}<2.25$ $\quad \therefore 2.13<\sqrt{5}<2.25$ $x_4=\frac{2.13+2.25}{2}=2.19$ $f(2.19)=(2.19)^2-5=-0.2039<0$ $\therefore 2.19<\sqrt{5}<2.25$ $\quad \therefore \sqrt{5}=2.2$ correct to one decimal place	2	**2 marks** • Correct solution using an appropriate number of iterations **1 mark** • Correctly applies the “halving the interval” method at least once
1 (b) (i) $\int 6x\sqrt{9-x^2}\,dx$ $\quad u=9-x^2,\ du=-2x\,dx$ $=-3\int\sqrt{u}\,du$ $=-3\times\frac{2}{3}u\sqrt{u}+c$ $=-2(9-x^2)\sqrt{9-x^2}+c$	3	**3 marks** • Correct solution using the given substitution **2 marks** • Correct primitive in terms of *u* **1 mark** • Correct integrand in terms of *u* • Correctly finds answer using an alternative approach
1 (b) (ii) $\int_0^{\frac{3}{2}}\frac{dx}{\sqrt{9-x^2}}=\int_0^{\frac{\pi}{6}}\frac{3\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}}$ $\quad x=3\sin\theta,\ dx=3\cos\theta\,d\theta$ $\quad$ when $x=0$, $\theta=0$; when $x=\frac{3}{2}$, $\theta=\frac{\pi}{6}$ $=\int_0^{\frac{\pi}{6}}\frac{3\cos\theta\,d\theta}{3\cos\theta}$ $=\int_0^{\frac{\pi}{6}}d\theta$ $=[\theta]_0^{\frac{\pi}{6}}$ $=\frac{\pi}{6}$	3	**3 marks** • Correct solution using the given substitution • *Note: solving as an indefinite integral, then using answer to find definite integral is acceptable* **2 marks** • Correct primitive in terms of θ • Correct integrand in terms of θ, including the correct limits **1 mark** • Correct integrand in terms of θ without the limits • Correctly finds answer using an alternative approach
1 (b) (iii) $\int\frac{-\sin 2x}{2+3\cos^2 x}dx$ $\quad u=2+3\cos^2 x,\ du=-6\cos x\sin x\,dx=-3\sin 2x\,dx$ $=\frac{1}{3}\int\frac{du}{u}$ $=\frac{1}{3}\ln u+c$ $=\frac{1}{3}\ln(2+3\cos^2 x)+c$	3	**3 marks** • Correct solution using the given substitution **2 marks** • Correct primitive in terms of *u* **1 mark** • Correct integrand in terms of *u* • Correctly finds answer using an alternative approach

<table>
<tr><th colspan="2">Solution</th><th>Marks</th><th>Comments</th></tr>
<tr><td colspan="4">QUESTION 2</td></tr>
<tr><td>2(a) (i)</td><td>The tangent at $x = 0$ would cut the x-axis close to $x = -2$, which would be a good approximation to one of the negative solutions, but not the positive solution.
This occurred because $x = 0$ is on the opposite side of the stationary point to the positive solution.</td><td>1</td><td>1 mark
• Valid explanation</td></tr>
<tr><td>2(a) (ii)</td><td>The approximation would fail at either stationary point, as the tangent would be horizontal meaning it will never cut the x-axis.
In addition, if it is a stationary point then the derivative is zero at this point so zero would be substituted into the denominator of Newton's formula, thus making it undefined.
Stationary points are $x = 2$ and $x = -4$</td><td>2</td><td>2 marks
• Locate a correct example with a valid explanation
1 mark
• Locates a correct example</td></tr>
<tr><td>2 (a) (iii)</td><td>$f(x) = x^3 + 3x^2 - 24x - 40 \qquad f'(x) = 3x^2 + 6x - 24$
$x_0 = 4$
$x_1 = 4 - \frac{f(4)}{f'(4)} = 4 - \frac{-24}{48} = 4.5$
$x_2 = 4.5 - \frac{f(4.5)}{f'(4.5)} = 4.5 - \frac{3.875}{63.75} = 4.44$
$x_3 = 4.44 - \frac{f(4.44)}{f'(4.44)} = 4.44 - \frac{0.109184}{61.7808} = 4.44$
$\therefore x = 4.44$ is the positive solution, correct to two decimal places</td><td>3</td><td>3 marks
• Correctly finds the approximation to the positive solution
2 marks
• Finds one of the negative solutions
• Correctly uses Newton's Method at least once
1 mark
• Attempts to apply Newton's Method by using a correct formula</td></tr>
<tr><td>2 (b)</td><td>$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = e^{-x}$
$\frac{1}{2}v^2 = -e^{-x} + c$
$v^2 = -2e^{-x} + c$
when
$x = 0, v = 2$
$4 = -2e^0 + c$
$c = 6$
$v^2 = 6 - 2e^{-x}$
OR
$v\frac{dv}{dx} = e^{-x}$
$\int_2^v v\,dv = \int_0^x e^{-x}\,dx$
$\left[\frac{1}{2}v^2\right]_2^v = \left[-e^{-x}\right]_0^x$
$\frac{1}{2}v^2 - 2 = -e^{-x} + 1$
$\frac{1}{2}v^2 = -e^{-x} + 3$
$v^2 = 6 - 2e^{-x}$</td><td>3</td><td>3 marks
• Correct solution
2 marks
• Finds an expression for v^2 using a correct method
1 mark
• Identifies that $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ or equivalent expression linking velocity and displacement</td></tr>
</table>

Solution	Marks	Comments
QUESTION 3		
3 (a) (i) $v^2 = 36 - 6x - 2x^2$ $\frac{1}{2}v^2 = 72 - 12x - 4x^2$ $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $= -12 - 8x$ $= -8\left(x + \frac{3}{2}\right)$ $\therefore\ \ddot{x} = -n^2 X$, where $n = 2\sqrt{2}$ and $X = x + \frac{3}{2}$	**2**	**2 marks** • Correct solution **1 mark** • Identifies the condition for SHM • Finds an expression for acceleration in terms of displacement
3 (a) (ii) $v^2 \geq 0$ $36 - 6x - 2x^2 \geq 0$ $x^2 + 3x - 18 \leq 0$ $(x+6)(x-3) \leq 0$ $-6 \leq x \leq 3$ The particle oscillates between $x = -6$ and $x = 3$	**2**	**2 marks** • Correct solution **1 mark** • Progress towards solution using valid methods
3 (b) (i) $\ddot{x} = 0$ $\dot{x} = c_1$ when $t = 0$, $\dot{x} = 40$ $\therefore 40 = c_1$ $\dot{x} = 40$ $x = 40t + c_2$ when $t = 0$, $x = 0$ $\therefore\ 0 = 0 + c_2$ $c_2 = 0$ $x = 40t$ $\ddot{y} = -10$ $\dot{y} = -10t + c_3$ when $t = 0$, $\dot{y} = 0$ $\therefore\ 0 = 0 + c_3$ $c_3 = 0$ $\dot{y} = -10t$ $y = -5t^2 + c_4$ when $t = 0$, $y = 0$ $\therefore\ 0 = 0 + c_4$ $c_4 = 0$ $y = -5t^2$	**3**	**3 marks** • Correct solution **2 marks** • Correctly proves equation of motion for either x or y • Finds both equations without explicitly finding the value of the constants. **1 mark** • Finds equations for both $\dot{x}$ and $\dot{y}$
3 (b) (ii) Object hits the water when $y = -40$ i.e. $-5t^2 = -40$ $t^2 = 8$ $t = 2\sqrt{2}$ When $t = 2\sqrt{2}$; $x = 40(2\sqrt{2})$ $= 80\sqrt{2}$ Object hits the water after $2\sqrt{2}$ seconds, $80\sqrt{2}$ metres from the base of the cliff.	**2**	**2 marks** • Correct solution **1 mark** • Finds when it hits the water • Finds where it hits the water.
3 (b) (iii) When $t = 2\sqrt{2}$; $\dot{x} = 40$ and $\dot{y} = -10(2\sqrt{2})$ $= -20\sqrt{2}$  $V^2 = 40^2 + (20\sqrt{2})^2$ $= 2400$ $V = \sqrt{2400}$ $= 20\sqrt{6}$ $\tan\alpha = \frac{20\sqrt{2}}{40}$ $= \frac{1}{\sqrt{2}}$ $\alpha = 35.264.....$ $\therefore$ object hits the water at a speed of $20\sqrt{6}$ m/s at an angle of 35° to the water	**3**	**3 marks** • Correct solution *Note: angle can be either acute or obtuse* **2 marks** • Finds the velocity of the object • Finds the angle the object makes with the water (either acute or obtuse) **1 mark** • Calculates the horizontal and vertical components of the velocity *Note: correct answers based upon time found in part (ii) should be marked correct.*

Solution	Marks	Comments
QUESTION 4		
4 (a) (i) $x = A\cos(2t+\beta)$ $\dot{x} = -2A\sin(2t+\beta)$ $\ddot{x} = -4A\cos(2t+\beta)$ $= -4x$ Thus $x = A\cos(2t+\beta)$ is a possible equation of motion.	2	**2 marks** • Correct solution **1 mark** • Attempts to find acceleration as a function of time
4 (a) (ii) when $t = 0$, $x = 4$, $v = 2$; $x = A\cos(2t+\beta)$ $\quad$ $\dot{x} = -2A\sin(2t+\beta)$ $4 = A\cos\beta$ $\quad$ $2 = 2A\sin\beta$ $16 = A^2\cos^2\beta$ $\quad$ $4 = 4A^2\sin^2\beta$ $1 = A^2\sin^2\beta$ $A^2\cos^2\beta + A^2\sin^2\beta = 17$ $A^2 = 17$ $A = \sqrt{17}$ $\therefore$ amplitude of the motion is $\sqrt{17}$ metres	2	**2 marks** • Successfully shows result **1 mark** • Uses initial conditions in a valid attempt to show the given result
4 (a) (iii) $\dot{x} = -2A\sin(2t+\beta)$ $\therefore$ maximum speed of the particle is $2A = 2\sqrt{17}$ m/s	1	**1 mark** • Correct answer
4 (b) (i) $x = Vt\cos\theta$ when $x = R$; $R = Vt\cos\theta$ $t = \dfrac{R}{V\cos\theta}$ $y = Vt\sin\theta - \dfrac{1}{gt^2}$ when $y = 0$; $0 = V\left(\dfrac{R}{V\cos\theta}\right)\sin\theta - \dfrac{1}{2}g\left(\dfrac{R}{V\cos\theta}\right)^2$ $0 = \dfrac{R\sin\theta}{\cos\theta} - \dfrac{gR^2}{2V^2\cos^2\theta}$ $0 = 2V^2R\sin\theta\cos\theta - gR^2$ $0 = R\left(2V^2\sin\theta\cos\theta - gR\right)$ $R = 0$ or $R = \dfrac{2V^2\sin\theta\cos\theta}{g}$ $= \dfrac{V^2\sin 2\theta}{g}$ But $R \neq 0$, $\therefore$ $R = \dfrac{V^2\sin 2\theta}{g}$	2	**2 marks** • Correct solution **1 mark** • Eliminates t from the parametric equations
4 (b) (ii) $\dfrac{V^2}{g}$ is constant, thus R is a maximum when $\sin 2\theta$ is a maximum maximum $\sin 2\theta$ occurs when $2\theta = 90°$ i.e. $\theta = 45°$	1	**1 mark** • Correct explanation
4 (b) (iii) $\dfrac{dR}{dt} = \dfrac{dR}{d\theta} \times \dfrac{d\theta}{dt}$ $= \dfrac{2V^2\cos 2\theta}{g} \times k$ $= \dfrac{2kV^2\cos 2\theta}{g}$	2	**2 marks** • Correct solution **1 mark** • Finds $\dfrac{dR}{d\theta}$
4 (b) (iv) $\dfrac{d^2R}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dR}{dt}\right)$ $= \dfrac{d}{d\theta}\left(\dfrac{dR}{dt}\right) \times \dfrac{d\theta}{dt}$ $= -\dfrac{4kV^2\sin 2\theta}{g} \times k$ $= -\dfrac{4k^2V^2\sin 2\theta}{g}$ $= -4k^2R$ $\therefore$ motion is SHM as $\ddot{x} = -n^2x$, where $n = 2k$	2	**2 marks** • Correct solution **1 mark** • Finds $\dfrac{d^2R}{dt^2}$ in terms of θ